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FLEXIBILITY MATRIX FOR SKEW-CURVED BEAMS

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Abstract—In this paper, based on the principle of virtual work, the formulation of the flexibility matrix and the static analysis of a skew-curved beam in the most general case of loading and response are presented. Each differential element of the centroidal axis of the beam is given six degrees of freedom; namely, three translations and three rotations. Three internal forces and three internal moments are assumed to act at each point of the centroidal axis of the beam. Finally, the results of the method are illustrated through the derivation of the flexibility influence functions, associated with a cantilever helicoidal beam.

NOTATION

A123	Frenet trihedron
$\boldsymbol{b} = [b_1, b_2, b_3]^T$	unit binormal vector
D	matrix of direction cosines of $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$
$\mathbf{D}$	matrix defined in eqn (27)
F	flexibility matrix defined in eqn (22)
$F_{11}, F_{12}, F_{21}, F_{22}$	submatrices defined in eqn (22)
$H^\xi, \bar{H}^\xi$	matrices defined in eqns (11)
I	unit matrix
$I(\xi)$	relation defined in eqns (9) and (10)
$\boldsymbol{m} = [m_1, m_2, m_3]^T$	vector of continuous external moments
$\mathcal{M} = [\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3]^T$	vector of concentrated external moments
$\mathrm{M}_j = 1$	unit virtual moments
$\mathbf{M} = [M_1, M_2, M_3]^T$	vector of internal moments
$\boldsymbol{n} = [n_1, n_2, n_3]^T$	unit normal vector
$N^\xi, \bar{N}^\xi$	matrices defined in eqns (11)
$O\overset{*}{x}\overset{*}{y}\overset{*}{z}$	global coordinate system
$\mathrm{P}_j = 1$	unit virtual forces
$\boldsymbol{p}$	vector defined in eqns (25)
$\mathbf{q} = [q_1, q_2, q_3]^T$	vector of concentrated external forces.
$\boldsymbol{r} = [x, y, z]^T$	position vector
$\mathcal{R} = [\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3]^T$	vector of external concentrated forces
R^κ	matrix defined in eqn (6)
s	arc length
S	vector defined in eqn (36)
S	stiffness matrix
$\mathbf{T} = [T_1, T_2, T_3]^T$	vector of internal forces
$\boldsymbol{t} = [t_1, t_2, t_3]^T$	unit tangent vector
u	arbitrary parameter
$\boldsymbol{w}$	vector defined in eqns (26)
y_{kj}, y_{kl}	flexibility influence functions
$\mathbf{Y} = [y_1, y_2, y_3]^T$	vector of deflections
$\bar{\mathbf{Y}}^{\lambda\kappa}$	vector defined in eqn (19)
A, B	matrices defined in eqns (16), (17)
$\bar{\mathbf{A}}$	matrix defined in eqn (21)
$a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3, \beta_1, \beta_2, \beta_3$	constants given in the text
ψ_{ij}, ψ_{il}	flexibility influence functions
$\mathbf{\Psi} = [\psi_1, \psi_2, \psi_3]^T$	vector of rotations
$\bar{\mathbf{\Psi}}^{\lambda\kappa}$	vector defined in eqn (18)
κ, ξ	superscripts or arbitrary parameters
i, j, k, l	subscripts
λ, T	superscripts
r, σ	summation indices
*	star indicating quantities referred to the global coordinate system

1. INTRODUCTION

The analysis of skew-curved beams under static loading has been treated quite extensively by many investigators. The stiffness approach has been used by Baron and Michalos[1], Michalos[2] and Veltuni[3] in analysing continuous curved beams. Others, such as Fickel[4], Patel[5] and Pippard[6] have investigated continuous curved beams using the flexibility method. Tuma and Reddy[7] have applied the flexibility method in analysing laterally loaded planar continuous curved and bent beams. Also of interest is the work of Wittfoht[8] and Dabrowski[9], both of whom investigated horizontally curved beams. Dabrowski's analysis, like the analysis of Unold[10], was based upon the solution of the governing differential equations. Also, Abdul and Shukair[11] have investigated indeterminate continuous helicoidal girders by the flexibility approach, on the basis of the assumption that the work of shearing and normal forces may be neglected in the expression for the elastic-strain energy. Finally, in Refs.[12]–[18], static analyses of planar or skew-curved beams with constant radii of curvature and torsion were presented.

A more complete research of the aforegoing problem is included in the papers by Young[19] and Morris[20]. In the first, the flexibility method was used to determine the flexibility influence functions of a skew-curved beam, whose centroidal axis was considered to be composed by straight components. Limiting the deriving difference equation the aforegoing functions result for the cases of planar curved beams. In the second[20], the stiffness-matrix method was also used for the static analysis of planar curved beams with constant radius of curvature, by making the assumption that in the elastic–strain energy expression the influences of shearing deformation may be neglected. In Ref. [21] the closed form solution of the differential equations with variable coefficients, (in terms of generalized forces and displacements), governing the equilibrium of a skew-curved element is presented. Finally, Washizu[22] presents an approximate method of a cantilever skew-curved beam, with emphasis on the derivation of governing equations which take into account the effects of torsion and transverse shear deformation.

In this investigation, based on the principle of virtual work, the formulation of flexibility matrix and the static analysis of a skew-curved beam in the most general case of loading and response are presented. The beam was assumed to undergo infinitesimal deformations and therefore the entire development was based on linear elastic analysis. Each differential element of the centroidal axis of the beam was given six degrees of freedom; namely, three translations and three rotations. It must be noticed also that, applying the principle of virtual work, all the influences of internal forces and moments are taken into account. In addition, the flexibility matrix was determined with respect either to a local or to a global rectangular coordinate system. Finally, the results of the method were illustrated through the derivation of the flexibility influence functions, associated with a cantilever helicoidal beam. The derived relations are in agreement with those of Refs. [11] and [21].

2. MATHEMATICAL FORMULATION

Consider a skew-curved beam of uniform cross section, whose centroidal axis is defined by the position vector $\overset{*}{r} = \overset{*}{r}(u) = [\overset{*}{x}(u), \overset{*}{y}(u), \overset{*}{z}(u)]^T$; where u is an arbitrary parameter and the superscript "T" indicates the transpose of the vector. Consider also, at an arbitrary point A^κ of the beam, the Frenet tihedron A^κ 123 with unit vectors (t^κ, n^κ, b^κ); where $t^\kappa = [t_1{}^\kappa(u), t_2{}^\kappa(u), t_3{}^\kappa(u)]^T$ is the unit tangent vector pointing to the direction of increasing arc s, $n^\kappa = [n_1{}^\kappa(u), n_2{}^\kappa(u), n_3{}^\kappa(u)]^T$ is the unit normal vector pointing to the center of curvature and $b^\kappa = [b_1{}^\kappa(u), b_2{}^\kappa(u), b_3{}^\kappa(u)]^T$ is the unit binormal vector; b^κ is defined in such a way that the correspodning Frenet trihedron to be a right-handed system. Also, $\mathbf{T}^\kappa = [T_1{}^\kappa, T_2{}^\kappa, T_3{}^\kappa]^T$ and $\mathbf{M}^\kappa = [M_1{}^\kappa, M_2{}^\kappa, M_3{}^\kappa]^T$ denote the vectors of generalized forces (internal forces and moments respectively), while $\boldsymbol{\Psi}^\kappa = [\psi_1{}^\kappa, \psi_2{}^\kappa, \psi_3{}^\kappa]^T$ and $\mathbf{Y}^\kappa = [y_1{}^\kappa, y_2{}^\kappa, y_3{}^\kappa]^T$ denote the vectors of generalized displacements (rotations and translations respectively) of the cross section A^κ of the beam, Fig. 1 (1a, 1b). Note that the axes 1,2,3 of Frenet trihedron coincide with the principal axes of the cross section of the beam.

2.1. *Generalised forces*

Consider the cantilever skew-curved beam A^0B, at the point A^κ of which the concentrated external force $\mathcal{R}^\kappa = [P_1{}^\kappa, P_2{}^\kappa, P_3{}^\kappa]$ and moment $\mathcal{M}^\kappa = [M_1{}^\kappa, M_2{}^\kappa, M_3{}^\kappa]^T$ are applied, Fig. 2.

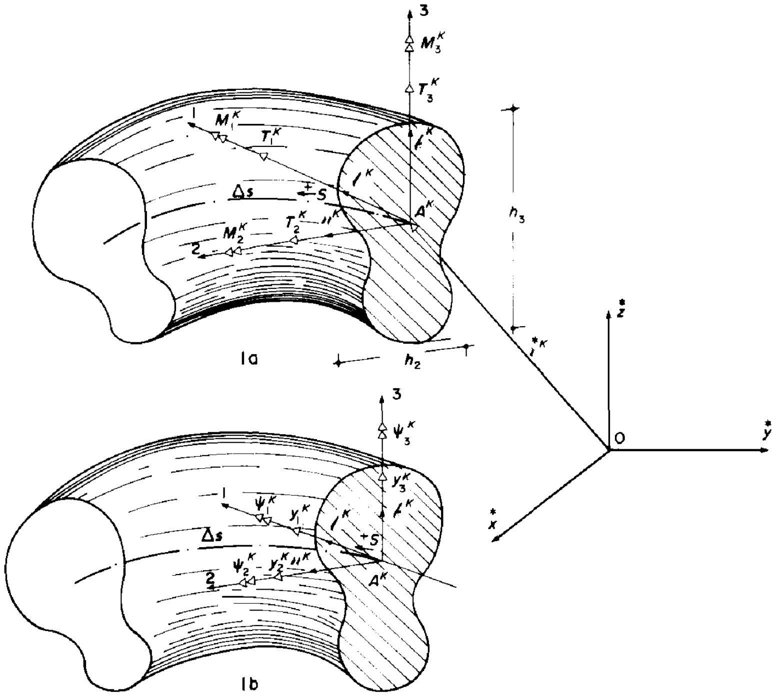
Fig. 1(a, b). Geometry and sign convention of an element of a skew-curved beam.

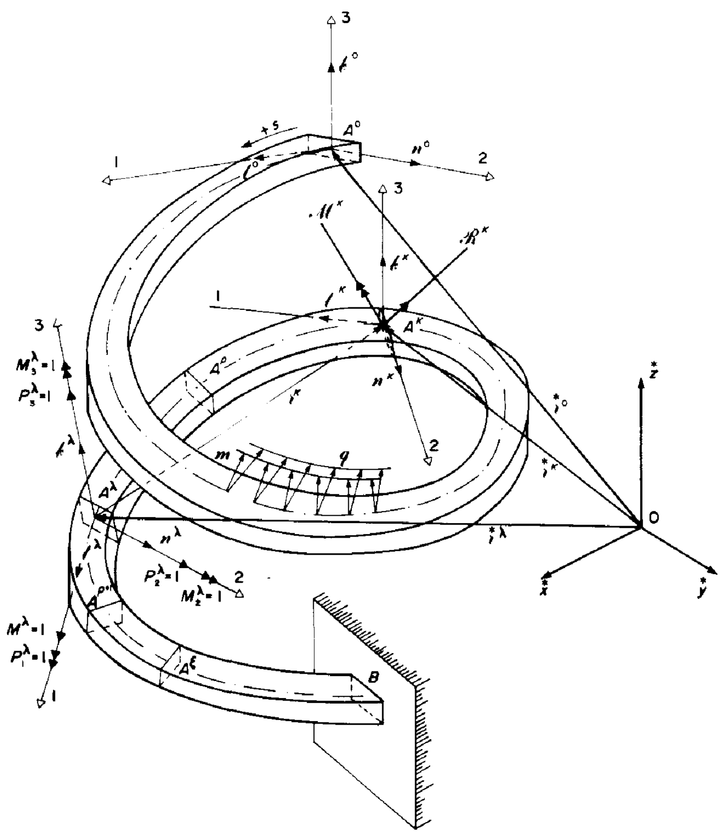
Fig. 2. A cantilever skew-curved beam subjected to a general loading and to three unit virtual forces and moments at the point A^{λ}.

Based on the vectorial equations governing the equilibrium of the arc A^0A^λ, the vectors of generalised forces at a cross section A^λ of the beam are expressed by

$$\mathbf{T}^\lambda = [\mathcal{R}^\kappa l^\lambda, \mathcal{R}^\kappa n^\lambda, \mathcal{R}^\kappa b^\lambda]^T \tag{1}$$

$$\mathbf{M}^\lambda = [\mathcal{M}^\kappa l^\kappa, \mathcal{M}^\kappa n^\lambda, \mathcal{M}^\kappa b^\lambda]^T + r^\kappa \times [\mathcal{R}^\kappa l^\lambda, \mathcal{R}^\kappa n^\lambda, \mathcal{R}^\kappa b^\lambda]^T \tag{2}$$

where:

$$r^\kappa = \mathrm{D}^\lambda(\overset{*}{r}{}^\kappa - \overset{*}{r}{}^\kappa - \overset{*}{r}{}^\lambda) = (x^\kappa, y^\kappa, z^\kappa)^T. \tag{3}$$

In eqn (3) r^κ is the position vector related to the Frenet trihedron $A^\kappa 123$, while $\mathrm{D}^\lambda = [t_i^\lambda, n_i^\lambda, b_i^\lambda]^T$ denotes a (3×3) matrix of the direction cosines of the unit vectors $(r^\lambda, n^\lambda, b^\lambda)$ at the point A^λ; it is valid that $\mathrm{D}[\mathrm{D}]^T = \mathrm{D}[\mathrm{D}]^{-1} = \mathbf{I}$; $\mathbf{I}$ being the (3×3) unit matrix.

Equations (1) and (2) may be written, after some simple manipulations,

$$\mathbf{T}^\lambda = \mathrm{D}^\lambda [\mathrm{D}^\kappa]^T \mathcal{R}^\kappa \tag{4}$$

$$\mathbf{M}^\lambda = \mathrm{D}^\lambda [\mathrm{D}^\kappa]^T \mathcal{M}^\kappa + \mathrm{R}^\kappa \mathrm{D}^\lambda [\mathrm{D}^\kappa]^T \mathcal{R}^\kappa \tag{5}$$

where:

$$\mathrm{R}^\kappa = \begin{bmatrix} 0 & -z^\kappa & y^\kappa \\ z^\kappa & 0 & -x^\kappa \\ -y^\kappa & x^\kappa & 0 \end{bmatrix}. \tag{6}$$

Assume that $n+1$ concentrated external forces $\mathcal{R}^\kappa$ and moments $\mathcal{M}^\kappa$ act at $n+1$ points $A^\kappa(\kappa = 0, 1, \ldots, n)$ of the cantilever beam $A^0\mathbf{B}$ respectively. The generalised forces and moments at the cross section A^λ are determined by:

$$\mathbf{T}^\lambda = \mathrm{D}^\lambda \sum_{\kappa=0}^{\rho} [\mathrm{D}^\kappa]^T \mathcal{R}^\kappa \tag{7}$$

$$\mathbf{M}^\lambda = \mathrm{D}^\lambda \sum_{\kappa=0}^{\rho} [\mathrm{D}^\kappa]^T \mathcal{M}^\kappa + \sum_{\kappa=0}^{\rho} \mathrm{R}^\kappa \mathrm{D}^\lambda [\mathrm{D}^\kappa]^T \mathcal{R}^\kappa \tag{8}$$

where

$$\rho < \lambda < \rho + 1.$$

Finally, in the case of a continuous external loading $q(u) = [q_1(u), q_2(u), q_3(u)]^T$ and $m(u) = [m_1(u), m_2(u), m_3(u)]^T$ acting on the beam $A^0\mathbf{B}$, the generalised forces at $A^\lambda(u)$ result, (Fig. 2):

$$\mathbf{T}^\lambda = \mathrm{D}^\lambda \int_{u_0}^{u_\lambda} [\mathrm{D}^\xi]^T q(\xi) I(\xi)\, \mathrm{d}\xi \tag{9}$$

$$\mathbf{M}^\lambda = \mathrm{D}^\lambda \int_{u_0}^{u_\lambda} [\mathrm{D}^\xi]^T m(\xi) I(\xi)\, \mathrm{d}\xi + \int_{u_0}^{u_\lambda} \mathrm{R}^\xi \mathrm{D}^\lambda [\mathrm{D}^\xi]^T q(\xi) I(\xi)\, \mathrm{d}\xi \tag{10}$$

where the variable ξ takes the same values with the parameter u and $I(\xi) = (\mathrm{d}^2x/\mathrm{d}\xi^2 + \mathrm{d}^2y/\mathrm{d}\xi^2 + \mathrm{d}^2z/\mathrm{d}\xi^2)^{1/2}$.

2.2. *Generalized displacements*

Consider the cantilever skew-curved beam $A^0\mathbf{B}$, at the point A^κ of which the concentrated external force $\mathcal{R}^\kappa$ and moment $\mathcal{M}^\kappa$ are applied, Fig. 2. The determination of generalized displacements at a cross section A^λ will be achieved through the application of the *principle of* *virtual work*. Thus, for a random point A^ξ of the beam the matrices $\mathbf{H}^\xi$, $\bar{\mathbf{H}}^\xi$, $\mathbf{N}^\xi$ and $\bar{\mathbf{N}}^\xi$ are determined by:

$$\begin{aligned}
\mathbf{H}^\xi &= [\mathrm{H}^\xi_{ij}] = \mathrm{D}^\xi[\mathrm{D}^\lambda]^T \\
\bar{\mathbf{H}}^\xi &= [\bar{\mathrm{H}}^\xi_{ij}] = \mathrm{R}^\lambda \mathrm{D}^\xi[\mathrm{D}^\lambda]^T \quad i,j = 1,2,3 \\
\mathbf{N}^\xi &= [\mathrm{N}^\xi_{ij}] = \mathrm{D}^\xi[\mathrm{D}^\kappa]^T \\
\bar{\mathbf{N}}^\xi &= [\bar{\mathrm{N}}^\xi_{ij}] = \mathrm{R}^\kappa \mathrm{D}^\xi[\mathrm{D}^\kappa]^T.
\end{aligned} \tag{11}$$

Using relations [4], (6), and (11) the vectors of generalized forces at A^ξ can be obtained:

$$\begin{aligned}
\mathbf{T}^\xi &= \mathbf{N}^\xi \mathcal{R}^\kappa \\
\mathbf{M}^\xi &= \mathbf{N}^\xi \mathcal{M}^\kappa + \bar{\mathbf{N}}^\xi \mathcal{R}^\kappa.
\end{aligned} \tag{12}$$

Furthermore, the unit virtual forces $\mathrm{P}_j^\lambda = 1$ and moments $\mathrm{M}_j^\lambda = 1$ are successively applied at A^λ, Fig. 2.

The vectors of generalized forces at A^ξ, due to these virtual loadings, result:

$$\begin{aligned}
\mathbf{M}^\xi_{\mathrm{M}_j} &= [\mathrm{H}^\xi_{1j}, \mathrm{H}^\xi_{2j}, \mathrm{H}^\xi_{3j}] \quad \text{due to} \quad \mathrm{M}_j^\lambda = 1 \\
\mathbf{T}^\xi_{\mathrm{P}_j} &= [\mathrm{H}^\xi_{1j}, \mathrm{H}^\xi_{2j}, \mathrm{H}^\xi_{3j}] \quad \text{due to} \quad \mathrm{P}_j^\lambda = 1 \\
\mathbf{M}^\xi_{\mathrm{P}_j} &= [\bar{\mathrm{H}}^\xi_{1j}, \bar{\mathrm{H}}^\xi_{2j}, \bar{\mathrm{H}}^\xi_{3j}]
\end{aligned} \tag{13}$$

Using the principle of virtual work and relations (11) to (13) the components of generalized displacements at A^λ are given by:

$$\psi_j^\lambda = \alpha_r \int_{u_\lambda}^{u_B} \mathrm{H}^\xi_{rj}\{\mathrm{N}^\xi_{r\sigma}\mathrm{M}^\kappa_\sigma + \bar{\mathrm{N}}^\xi_{r\sigma}\mathrm{P}_\sigma{}^\kappa\} I(\xi)\,\mathrm{d}\xi \tag{14}$$

$$y_j^\lambda = \beta_r \int_{u_\lambda}^{u_B} \mathrm{H}^\xi_{rj}\mathrm{N}^\xi_{r\sigma}\mathrm{P}_\sigma{}^\kappa I(\xi)\,\mathrm{d}\xi + \alpha_r \int_{u_\lambda}^{u_B} \bar{\mathrm{H}}^\xi_{rj}\{\mathrm{N}^\xi_{r\sigma}\mathrm{M}_\sigma{}^\kappa + \bar{\mathrm{N}}^\xi_{r\sigma}\mathrm{P}_\sigma{}^\kappa\} I(\xi)\,\mathrm{d}\xi \tag{15}$$

where $j = 1,2,3$ and r, σ are summation indices $1,2,3$, while $a_1, a_2, a_3, \beta_1, \beta_2, \beta_3$ are constants given by:

$$a_1 = \frac{1}{\mathrm{GI}_1}, a_2 = \frac{1}{\mathrm{EI}_2}, a_3 = \frac{1}{\mathrm{EI}_3}, \beta_1 = \frac{1}{\mathrm{EF}}, \beta_2 = \frac{1}{\mathrm{G}\overset{*}{\mathrm{F}}_2}, \beta_3 = \frac{1}{\mathrm{G}\overset{*}{\mathrm{F}}_3}, \overset{*}{\mathrm{F}}_2 = \lambda_2\mathrm{F}, \overset{*}{\mathrm{F}}_3 = \lambda_3\mathrm{F}.$$

In these relations E, G denote the elastic and shear moduli respectively; I_1 is the torsional moment of inertia; I_2 and I_3 are the moments of inertia of the cross section about the axes of $\boldsymbol{n}$ and $\boldsymbol{b}$ respectively; F is the area of the cross section. Finally, λ_2 and λ_3 are coefficients depending on the shape of the cross section.

The vectors of generalized displacements at A^λ through eqns (14) and (15), derive as:

$$\mathbf{\Psi}^\lambda = \int_{u_\lambda}^{u_B} [\mathbf{A}\mathbf{H}^\xi]^T\{\mathbf{N}^\xi\mathcal{M}^\kappa + \bar{\mathbf{N}}^\xi\mathcal{R}^\kappa\} I(\xi)\,\mathrm{d}\xi \tag{16}$$

$$\mathbf{Y}^\lambda = \int_{u_\lambda}^{u_B} [\mathbf{B}\mathbf{H}^\xi]^T\mathbf{N}^\xi\mathcal{R}^\kappa I(\xi)\,\mathrm{d}\xi + \int_{u_\lambda}^{u_B} [\mathbf{A}\bar{\mathbf{H}}^\xi]^T\{\mathbf{N}^\xi\mathcal{M}^\kappa + \bar{\mathbf{N}}^\xi\mathcal{R}^\kappa\} I(\xi)\,\mathrm{d}\xi \tag{17}$$

where the elements of the diagonal matrices **A** and **B** of dimensions (3×3), are the ceofficients a_r and β_r respectively.

When $n+1$ concentrated external loadings $\mathcal{R}^\kappa$ and $\mathcal{M}^\kappa$ are applied at $n+1$ points $\mathbf{A}^\kappa$ $(\kappa = 0,1,2,\ldots,n)$ of the cantilever beam $\mathbf{A}^0\mathbf{B}$ respectively, the vectors of generalized displacements $\bar{\Psi}^{\lambda}$ and $\bar{Y}^{\lambda}$ at A^{λ} result:

$$\bar{\Psi}^{\lambda} = \sum_{\kappa=0}^{n} \Psi^{\lambda,\kappa} \tag{18}$$

$$\bar{Y}^{\lambda} = \sum_{\kappa=0}^{n} Y^{\lambda,\kappa}. \tag{19}$$

In relations (18) and (19) $\Psi^{\lambda,\kappa}$ and $Y^{\lambda,\kappa}$ are the vectors of generalized displacements at the cross section A^{λ} due to the concentrated loading at the point A^{κ}.

Finally, in the case when a continuous external load q and m and temperature changes are applied on the beam A^0B, the vectors of generalized displacements at A^{λ} result:

$$\Psi^{\lambda} = \int_{u_\lambda}^{u_B} [AH^{\xi}]^T \left\{ \int_{u_0}^{\xi} \{N^{\kappa} m(\kappa) + \bar{N}^{\kappa} q(\kappa)\} I(\kappa)\, d\kappa \right\} I(\xi)\, d\xi + \int_{u_\lambda}^{u_B} [\bar{A}H^{\xi}]^T \left\{ \int_{u_0}^{\xi} I(\kappa)\, d\kappa \right\} I(\xi)\, d\xi \tag{20}$$

$$\begin{aligned} Y^{\lambda} = {} & \int_{u_\lambda}^{u_B} [BH^{\xi}]^T \left\{ \int_{u_0}^{\xi} N^{\kappa} q(\kappa) I(\kappa)\, d\kappa \right\} I(\xi)\, d\xi + \int_{u_\lambda}^{u_B} [A\bar{H}^{\xi}]^T \left\{ \int_{u_0}^{\xi} \{N^{\kappa} m(\kappa) \right. \\ & \left. + \bar{N}^{\kappa} q(\kappa)\} I(\kappa)\, d\kappa \right\} I(\xi)\, d\xi + \int_{u_\lambda}^{u_B} [\bar{A}\bar{H}^{\xi}]^T \left\{ \int_{u_0}^{\xi} I(\kappa)\, d\kappa \right\} I(\xi)\, d\xi. \end{aligned} \tag{21}$$

The elements of the diagonal matrix $\bar{A}$ of dimensions (3×3) appearing in the eqns (20) and (21) are the coefficients $\bar{a}_1 = \lambda_0(\Delta t_3/h_3)$, $\bar{a}_2 = (\Delta t_2/h_2)$ and $\bar{a}_3 = \lambda_0 t_s$ where $\Delta t_3 = t_r - t_l$ and $\Delta t_2 = t_0 - t_u$ represent constant temperature differences of the limits of the cross section with respect to the b and n-axes respectively; t_s represents the uniform change of temperature of the centroidal axis and λ_0 the coefficient of the thermal expansion; Finally h_3 and h_2 denote the maximum dimensions of the cross section parallel to the axes of b and n respectively, Fig. 1.

3. FLEXIBILITY AND STIFFNESS MATRICES

Assume the cantilever skew-curved beam **AB** and the Frenet trihedron **A*tnb*** at the free end **A**. We notice the axes of the system by the numbers 1, 2, 3, and 4, 5, 6, in such a way that 1 and 4 correspond to the vector t; 2 and 5 correspond to the vector n and 3 and 6 to the vector b. In the following, symbols representing moments or rotations will be defined by the subscripts i and j $(i, j = 1, 2, 3)$, while symbols representing forces or translations by the subscripts k and l $(k, l = 4, 5, 6)$ respectively, Fig. 3(a). By applying the principle of virtual work for unit loadings acting at the free end **A** and based on relations (14) and (15), the (6×6)-flexibility matrix **F** of the beam **AB** is determined and given by:

$$\mathbf{F} = \left[\begin{array}{c|c} [\psi_{ij}] & [\psi_{il}] \\ \hline [y_{kj}] & [y_{kl}] \end{array}\right] = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}. \tag{22}$$

Thus, physically, ψ_{ij} is the rotation about the i-axis due to a unit moment about the j-axis; y_{kl} is the translation in the direction k due to unit force in the l-direction, Fig. 3(b). The analytical expressions of the flexibility influence functions ψ_{ij}, ψ_{il}, y_{kj} and y_{kl} are given in the Appendix. It must be noticed that, due to the *Maxwell's reciprocal theorem*, the following relations are valid:

$$\begin{aligned} \psi_{ij} &= \psi_{ji} \\ y_{kl} &= y_{lk} \\ \psi_{il} &= y_{kj}. \end{aligned} \tag{23}$$

Consequently **F** is a symmetrical matrix and the (3×3) submatrices F_{11}, F_{12}, F_{21} and F_{22} satisfy the relations: F_{11}, F_{22} = symmetrical submatrices

$$F_{12} = [F_{21}]^T \tag{24}$$

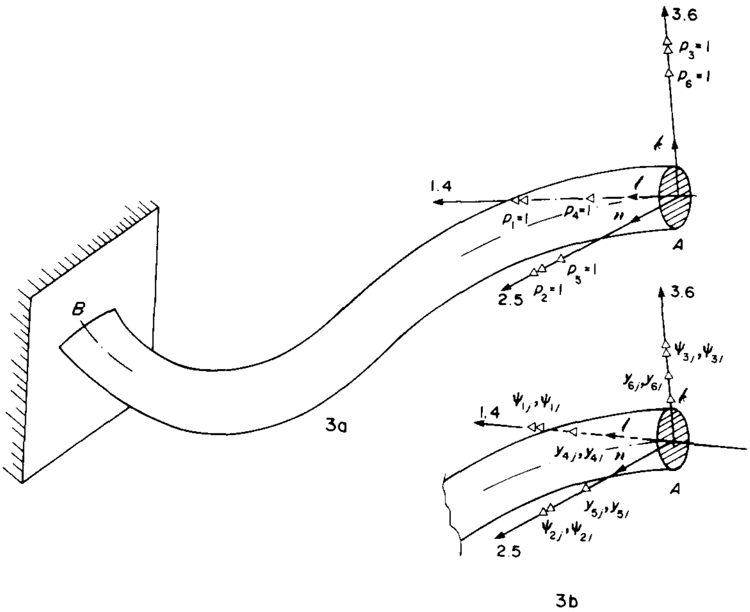

Fig. 3. (a), A cantilever skew-curved beam subjected to a unit virtual loading at its free end; (b), A physical interpretation of the flexibility influence functions of the cantilever beam.

The inverse of F is the stiffness matrix S of the beam, i.e.

$$\mathbf{S} = \mathbf{F}^{-1}.$$

The matrices F and S are associated with the local system A123.

Their transformation concerning to the global coordinate system $\overset{*}{o}\overset{*}{x}\overset{*}{y}\overset{*}{z}$ can be achieved through the following equations:

$$r = \mathrm{D}\overset{*}{r},\ w = \mathbf{F}p,\ p = \mathbf{S}w \tag{25}$$

$$w = \mathbf{D}\overset{*}{w},\ p = \mathbf{D}\overset{*}{p} \tag{26}$$

where p is the vector of external moments and forces being applied at the free end A with respect to the axes of Frenet trihedron A123 ($p = [\mathcal{M}, \mathfrak{R}]^T = [p_1, \ldots, p_6]^T$); w is the vector of generalized displacements of the free end A with respect to the Frenet trihedron, due to the loading p ($w = [\mathbf{\Psi}, \mathbf{Y}]^T = [w_1, \ldots, w_6]^T$) and D is a (6×6) matrix of the form:

$$\mathbf{D} = \begin{bmatrix} \mathrm{D} & (0) \\ (0) & \mathrm{D} \end{bmatrix} \tag{27}$$

where (0) is the (3×3) zero matrix.

From relations (25)–(27) the following two equations result:

$$\overset{*}{\mathbf{F}} = \mathbf{D}^T\mathbf{F}\mathbf{D} \tag{28}$$

$$\overset{*}{\mathbf{S}} = \mathbf{D}^T\mathbf{S}\mathbf{D} \tag{29}$$

where $\overset{*}{\mathbf{F}}$ and $\overset{*}{\mathbf{S}}$ are the flexibility and stiffness matrices associated with the global coordinate system $\overset{*}{o}\overset{*}{x}\overset{*}{y}\overset{*}{z}$.

4. STATICALLY INDETERMINATE SKEW-CURVED BEAMS

In this section the static analysis of skew-curved beams of practical engineering importance is presented; the analysis is based on the method developed in the previous sections.

4.1 *Beam with both fixed ends*

Let it be that a skew-curved beam **AB** with both fixed ends is subjected to a general loading. Consider also the corresponding cantilever beam, with **B** its fixed end. The vector of generalized displacements w at the cross section **A** of the indeterminate beam **AB** is defined by:

$$w = w_0 + \mathbf{F}_p \tag{30}$$

where **F** is the flexibility matrix of the cantilever **AB**; $w_0 = [\mathbf{\Psi}_0, \mathbf{Y}_0]^T$ is the vector of generalized displacements of the cantilever **AB** at **A** due to the external loading and $p = [\mathbf{M}_0, \mathbf{T}_0]^T$ is the vector of the unknown reactions of the indeterminate beam **AB** at the end **A**. The kinematic boundary conditions of the cross section **A** are:

$$w = \mathbf{0}. \tag{31}$$

Inserting (31) to (30) the unknown vector p can be determined by:

$$p = -\mathbf{S}w_0. \tag{32}$$

The definition of the vectors of generalized forces and displacements at a generic point of the indeterminate beam **AB** can be achieved through relations (7)–(10) and (18)–(21).

4.2 *Beam with one end fully fixed and the other end supported by an immovable spherical hinge*

Consider the skew-curved beam **AB**, with one fixed end at the point **B** and one spherical hinge at the point **A**. Consider also, the corresponding cantilever beam **AB**. The static and kinematic boundary conditions of the indeterminate beam **AB** at the cross section **A** are:

$$\mathbf{M} = \mathbf{Y} = \mathbf{0}. \tag{33}$$

Using (33), relation (30) becomes:

$$\begin{bmatrix} \mathbf{\Psi} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathrm{F}_{11} & \mathrm{F}_{12} \\ \mathrm{F}_{21} & \mathrm{F}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ -\mathbf{T} \end{bmatrix} + \begin{bmatrix} \mathbf{\Psi}_0 \\ \mathbf{Y}_0 \end{bmatrix} \tag{34}$$

where $\mathbf{\Psi}_0$ and $\mathbf{Y}_0$ are the vectors of generalized displacements of the cantilever **AB** at **A**. Through relation (34) the unknown vectors **T** and $\mathbf{\Psi}$ can be determined by:

$$\begin{aligned} \mathbf{T} &= -\mathrm{F}_{22}^{-1}\mathbf{Y}_0 \\ \mathbf{\Psi} &= -\mathrm{F}_{12}\mathrm{F}_{22}^{-1}\mathbf{T} + \mathbf{\Psi}_0. \end{aligned} \tag{35}$$

The generalized forces and displacements at a generic point of the indeterminate beam **AB** result in the same way as in Section 4.1.

4.3 *Beam with one end fully fixed and the other end simply supported*

Assume the skew-curved beam **AB**, with one end **B** fully fixed and the other **A** simple supported.

If **S** denotes the unknown reaction at the point **A** the following relation is valid:

$$\mathbf{T} = \mathbf{S} = \|\mathbf{S}\|\hat{\mathbf{s}} \tag{36}$$

where $\|\mathbf{S}\|$ is the norm of the vector **S** and $\hat{\mathbf{s}}(s_1, s_2, s_3)$ is the known unit-vector of the direction of the support A. The static and kinematic boundary conditions at A are:

$$\begin{aligned} \mathbf{M} &= \mathbf{0} \\ \mathbf{Y}\hat{\mathbf{s}} &= 0. \end{aligned} \tag{37}$$

Using (37), (36) and (34) relation (30) is transformed to:

$$\mathbf{\Psi} = \mathrm{F}_{12}\mathbf{S} + \mathbf{\Psi}_0 \tag{38}$$

$$0 = [\hat{\mathbf{s}}]^T \mathrm{F}_{22}\mathbf{S} + [\hat{\mathbf{s}}]^T \mathbf{Y}_0 \tag{39}$$

and through eqn (39) the unknown norm derives:

$$\|\mathbf{S}\| = -\frac{s_r y_0^{\,r}}{s_r s_\sigma [\mathrm{F}_{22}^{\sigma r}]} \quad r, \sigma \text{ are summation indices } 1, 2, 3. \tag{40}$$

It should be noticed here, that all the previously derived vectors and matrices are related to the Frenet trihedron A123.

5. APPLICATION TO AN HELICOIDAL BEAM

The application chosen here presents the development of the complete set of flexibility influence functions for an helicoidal cantilever beam AB of uniform cross section and length $S = (\varphi/\theta)$, $(\theta = \sin\delta/\alpha)$. The parametric equations of the beam with respect to the global coordinate system $o\overset{*}{x}\overset{*}{y}\overset{*}{z}$ are:

$$\begin{aligned} \overset{*}{x} &= \alpha \cos\varphi \\ \overset{*}{y} &= \alpha \sin\varphi \\ \overset{*}{z} &= \zeta\varphi \\ \cot\delta &= \frac{\zeta}{\alpha} \end{aligned}$$

where α, φ are the polar coordinates of the corresponding circular cylinder; ζ is a constant given by $\zeta = (\beta/2\pi)$, in which β denotes the step of the helix; δ is the constant angle of every point of the helix between the tangent and the generator.

Based on the procedure described previously and by applying relations (41), the flexibility influence functions of the beam AB derive as:

$$\psi_{11} = \alpha \sin\delta[a_1(\sin^2\delta \mathrm{I}_4 + \varphi\cos^2\delta\cot^2\delta + 2\cos^2\delta \mathrm{I}_2) + a_2\mathrm{I}_3 + a_3\cos^2\delta(\varphi + \mathrm{I}_4 - 2\mathrm{I}_2)]$$

$$\psi_{12} = \alpha[a_1(\sin^2\delta \mathrm{I}_5 + \cos^2\delta \mathrm{I}_1) - a_2\mathrm{I}_5 - a_3\cos^2\delta(\mathrm{I}_1 - \mathrm{I}_5)]$$

$$\psi_{13} = \alpha\cos\delta[a_1\{\sin^2\delta(\mathrm{I}_2 - \mathrm{I}_4) + \cos^2\delta(\varphi - \mathrm{I}_2)\} - a_2\mathrm{I}_3 + a_3\{\cos^2\delta(\mathrm{I}_2 - \mathrm{I}_4) + \sin^2\delta(\varphi - \mathrm{I}_2)\}]$$

$$\psi_{21} = \psi_{12}$$

$$\psi_{22} = \frac{\alpha}{\sin\delta}[a_1\sin^2\delta \mathrm{I}_3 + a_2\mathrm{I}_4 + a_3\cos^2\delta \mathrm{I}_3]$$

$$\psi_{23} = \alpha\cot\delta[a_1\sin^2\delta(\mathrm{I}_1 - \mathrm{I}_5) + a_2\mathrm{I}_5 - a_3(\cos^2\delta \mathrm{I}_5 + \sin^2 \mathrm{I}_1)]$$

$$\psi_{31} = \psi_{13}$$

$$\psi_{32} = \psi_{23}$$

$$\psi_{33} = \alpha\frac{\cos^2\delta}{\sin\delta}[a_1\sin^2\delta(\varphi + \mathrm{I}_4 - 2\mathrm{I}_2) + a_2\mathrm{I}_3 + a_3(\cos^2\delta \mathrm{I}_4 + \varphi\sin^2\delta\tan^2\delta + 2\sin^2\delta \mathrm{I}_2)]$$

$$\psi_{14}=\alpha^2\cos\delta[a_1\{\sin^2\delta(2I_2-2I_4-I_{10})+\cos^2\delta(2\varphi-2I_2-I_6)\}-a_2(I_3-I_{10}) + a_3\{\cos 2\delta(-\varphi+2I_2-I_4)+\cos^2\delta(I_6-I_{10})\}]$$

$$\psi_{15}=\alpha^2\cot\delta[a_1\{\sin^2\delta(I_9-I_5)+\cos^2\delta(I_7-I_1)+a_2I_8-a_3\{\sin^2\delta(I_1-I_5)+\cos^2\delta(I_7-I_9)\}]$$

$$\psi_{16}=\frac{\alpha^2}{\sin\delta}[a_1\{\sin^2\delta\cos 2\delta(I_4-I_2)+\cos^2\delta\cos 2\delta(-\varphi+I_2)+\cos^2\delta(\sin^2\delta I_{10}+\cos^2\delta I_6)\} - a_2(\cos^2\delta I_{10}+\sin^2\delta I_3)+a_3\sin\delta\cos\delta\{\sin 2\delta(-\varphi+2I_2-I_4)+\cot\delta\cos^2\delta(I_{10}-I_6)\}]$$

$$\psi_{24}=\alpha^2\cot\delta[a_1\sin^2\delta(2I_1-2I_5-I_8)+a_2(I_5-I_9)-a_3\{\cos 2\delta(I_5-I_1)+\cos^2\delta I_8\}]$$

$$\psi_{25}=\frac{\alpha^2\cot\delta}{\sin\delta}[a_1\sin^2\delta(I_{10}-I_3)-a_2I_{10}+a_3(\sin^2\delta I_3+\cos^2\delta I_{10})]$$

$$\psi_{26}=\alpha^2[a_1\{\cos 2\delta(I_5-I_1)+\cos^2\delta I_8\}+a_2(I_5+\cot^2\delta I_9)-a_3\cot\delta\{\sin 2\delta(I_5-I_1)-\cot\delta\cos^2\delta I_8\}]$$

$$\psi_{34}=\frac{\alpha^2\cos^2\delta}{\sin\delta}[a_1\sin^2\delta(2\varphi-4I_2-I_6+2I_4+I_{10})+a_2(I_3-I_{10}) + a_3\{\cos 2\delta(I_4-I_2)+\tan^2\delta\cos 2\delta(-\varphi+I_2)+\sin^2\delta I_6+\cos^2\delta I_{10}\}]$$

$$\psi_{35}=\alpha^2\cot^2\delta[a_1\sin^2\delta(I_7-I_9-I_1+I_5)-a_2I_8-a_3\{\tan^2\delta(\cos^2\delta I_5+\sin^2\delta I_1)+\cos^2\delta I_9+\sin^2\delta I_7\}]$$

$$\psi_{36}=\alpha^2\cos\delta\Big[a_1\{\cos 2\delta(2I_2-I_4-\varphi)+\cos^2\delta(I_6-I_{10})\}+a_2(I_3+\tan^2\delta I_{10})+ a_3\Big\{\frac{\sin^2 2\delta}{2\sin^2\delta}(I_4-I_2)+2\sin^2\delta(-\varphi+I_2)-\cot^2\delta(\cos^2\delta I_{10}+\sin^2\delta I_6)\Big\}\Big]$$

$$y_{41}=\psi_{14}$$

$$y_{42}=\psi_{24}$$

$$y_{43}=\psi_{34}$$

$$y_{51}=\psi_{15}$$

$$y_{52}=\psi_{25}$$

$$y_{53}=\psi_{35}$$

$$y_{61}=\psi_{16}$$

$$y_{62}=\psi_{26}$$

$$y_{63}=\psi_{36}$$

$$y_{44}=\frac{\alpha^3\cos^2\delta}{\sin\delta}\Big[a_1\sin^2\delta(4\varphi+4I_4+I_{13}-8I_2-4I_6+4I_{10})+a_2(I_3+I_{14}-2I_{10}) + a_3\Big\{\frac{\cos^2 2\delta}{\cos^2\delta}(\varphi+I_4-2I_2)+\cos^2\delta I_{13}+2\cos 2\delta(I_{10}-I_6)\Big\}\Big] + \alpha\sin\delta[\beta_1(\sin^2\delta I_4+\varphi\cos^2\delta\cot^2\delta+2\cos^2\delta I_2)+\beta_2I_3+\beta_3\cos^2\delta(\varphi+I_4-2I_2)]$$

$$y_{45}=\alpha^3\cot^2\delta[a_1\sin^2\delta(2I_7-2I_1-2I_9+2I_5-I_{15}+I_8)-a_2(I_8-I_{15}) - a_3\{\cos 2\delta\tan^2\delta(I_5-I_1)+\sin^2\delta I_8+\cos 2\delta(I_9-I_7)+\cos^2\delta I_{15}\}] + \alpha[\beta_1(\sin^2\delta I_5+\cos^2\delta I_1)-\beta_2I_5-\beta_3\cos^2\delta(I_1-I_5)]$$

$$y_{46}=\frac{\alpha^3\cot\delta}{\sin\delta}\Big[a_1\sin^2\delta\{\cos 2\delta(4I_2-2I_4-I_{10}+I_6-2\varphi)+\cos^2\delta(2I_6-2I_{10}-I_{13})\} + a_2\{\sin^2\delta(I_3-I_{10})+\cos^2\delta(I_{10}-I_{14}\}+a_3\Big\{\frac{\sin 4\delta}{2}\tan\delta(I_4-2I_2+\varphi) + \frac{\sin 2\delta}{2}(\sin 2\delta-\cot\delta\cos 2\delta)(I_{10}-I_6)-\cos^4\delta I_{13}\Big\}\Big] + \frac{\alpha\sin 2\delta}{2\sin\delta}[\beta_1\{\sin^2\delta(I_2-I_4)+\cos^2\delta(\varphi-I_2)\}-\beta_2I_3+\beta_3\{\cos^2\delta(I_2-I_4)+\sin^2\delta(\varphi-I_2)\}]$$

$$y_{54}=y_{45}$$

$$y_{55} = \frac{\alpha^3 \cot^2 \delta}{\sin \delta}[a_1 \sin^2 \delta(\mathrm{I}_{14} + \mathrm{I}_3 - 2\mathrm{I}_{10}) + a_2\mathrm{I}_{13} + a_3(\sin^2 \delta \tan^2 \delta \mathrm{I}_3 + \cos^2 \delta \mathrm{I}_{14} + 2 \sin^2 \delta \mathrm{I}_{10})]$$
$$+ \frac{\alpha}{\sin \delta}[\beta_1 \sin^2 \delta \mathrm{I}_3 + \beta_2 \mathrm{I}_4 + \beta_3 \cos^2 \delta \mathrm{I}_3]$$

$$y_{56} = \frac{\alpha^3 \cot \delta}{\sin \delta}[a_1 \sin \delta\{\cos 2\delta(\mathrm{I}_9 - \mathrm{I}_7 - \mathrm{I}_5 + \mathrm{I}_1) + \cos^2 \delta(\mathrm{I}_{15} - \mathrm{I}_8)\} - a_2(\sin \delta \mathrm{I}_8 + \cot \delta \cos \delta \mathrm{I}_{15})$$
$$- a_3\{2 \sin^3 \delta(\mathrm{I}_5 - \mathrm{I}_1) + \cos \delta \sin 2\delta(\mathrm{I}_9 - \mathrm{I}_7) - \cos^2 \delta \sin \delta(\mathrm{I}_8 + \cot^2 \delta \mathrm{I}_{15})\}]$$
$$+ \alpha \cot \delta[\beta_1 \sin^2 \delta(\mathrm{I}_1 - \mathrm{I}_5) + \beta_2 \mathrm{I}_5 - \beta_3(\sin^2 \delta \mathrm{I}_1 + \cos^2 \delta \mathrm{I}_5)]$$

$$y_{64} = y_{46}$$

$$y_{65} = y_{56}$$

$$y_{66} = \frac{\alpha^3}{\sin \delta}[a_1\{\cos^4 \delta \mathrm{I}_{13} + \cos^2 2\delta(\mathrm{I}_4 + \varphi - 2\mathrm{I}_2) + 2 \cos 2\delta \cos^2 \delta(\mathrm{I}_{10} - \mathrm{I}_6)\}$$
$$+ a_2\{\sin^2 \delta \mathrm{I}_3 + \cot^2 \delta \cos^2 \delta \mathrm{I}_{14} + 2 \cos^2 \delta \mathrm{I}_{10}\}$$
$$+ a_3\{\sin^2 2\delta(\varphi + \mathrm{I}_4 - 2\mathrm{I}_2) + \cot^2 \delta \cos^4 \delta \mathrm{I}_{13} - 4 \cos^4 \delta(\mathrm{I}_{10} - \mathrm{I}_6)\}]$$
$$+ \frac{\alpha \cos^2 \delta}{\sin \delta}[\beta_1 \sin^2 \delta(\varphi + \mathrm{I}_4 - 2\mathrm{I}_2) + \beta_2 \mathrm{I}_3 + \beta_3(\cos^2 \delta \mathrm{I}_4 + \varphi \sin^2 \delta \tan^2 \delta + 2 \sin^2 \delta \mathrm{I}_2)].$$

The integrals I_1–I_{15} are given in the appendix. In some special cases, the above given relations treated by the corresponding Refs. [11] and [21] are in agreement with the results derived in these references.

6. CONCLUSIONS

In this paper the formulation of the flexibility matrix and the static analysis of a skew-curved beam, in the most general case of loading and response are presented based on the principle of virtual work. Among the most important results of this investigation one may list the following:

1. The derivation of the vectors of generalized forces and displacements at a random point of a cantilever skew-curved beam under general loading by taking into consideration all the internal forces and moments.
2. The exact formulation of the flexibility matrix of a cantilever skew-curved beam with respect to a local or global coordinate system;
3. The static analysis of skew-curved beams in the most general case of loading and response by using the flexibility matrix, and
4. The derivation of flexibility influence functions associated with a cantilever helicoidal beam, by considering all the internal forces and moments acting on the beam for the expression of the principle of virtual work.

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APPENDIX

The analytical expressions for the flexibility influence functions included in relations (22) are given by:

$$\begin{aligned}
\psi_{ij} &= \alpha_r \int_{u_A}^{u_B} \mathrm{H}_{ri}^{\xi}\mathrm{H}_{rj}^{\xi}\, I(\xi)\,\mathrm{d}\xi \\
\psi_{il} &= \alpha_r \int_{u_A}^{u_B} \bar{\mathrm{H}}_{rl}^{\xi} I(\xi)\,\mathrm{d}\xi \\
y_{kj} &= \alpha_r \int_{u_A}^{u_B} \bar{\mathrm{H}}_{rk}^{\xi}\mathrm{H}_{rj}^{\xi} I(\xi)\,\mathrm{d}\xi \qquad i, j, k, l = 1, 2, 3 \\
y_{kl} &= \beta_r \int_{u_A}^{u_B} \mathrm{H}_{rk}^{\xi}\mathrm{H}_{rl}^{\xi} I(\xi)\,\mathrm{d}\xi + \alpha_r \int_{u_A}^{u_B} \bar{\mathrm{H}}_{rk}^{\xi}\bar{\mathrm{H}}_{rl}^{\xi} I(\xi)\,\mathrm{d}\xi
\end{aligned} \tag{41}$$

where r is summation index 1, 2, 3

The analytical expressions for the integrals I_1 to I_{15} are expressed by:

$$\mathrm{I}_1 = \int_0^{\varphi} \sin\varphi\,\mathrm{d}\varphi = -\cos\varphi + 1$$

$$\mathrm{I}_2 = \int_0^{\varphi} \cos\varphi\,\mathrm{d}\varphi = \sin\varphi$$

$$\mathrm{I}_3 = \int_0^{\varphi} \sin^2\varphi\,\mathrm{d}\varphi = \frac{\varphi}{2} - \frac{\sin 2\varphi}{4}$$

$$\mathrm{I}_4 = \int_0^{\varphi} \cos^2\varphi\,\mathrm{d}\varphi = \frac{\varphi}{2} + \frac{\sin 2\varphi}{4}$$

$$\mathrm{I}_5 = \int_0^{\varphi} \sin\varphi\cos\varphi\,\mathrm{d}\varphi = -\frac{1}{4}(\cos 2\varphi - 1)$$

$$\mathrm{I}_6 = \int_0^{\varphi} \varphi\sin\varphi\,\mathrm{d}\varphi = \sin\varphi - \varphi\cos\varphi$$

$$\mathrm{I}_7 = \int_0^{\varphi} \varphi\cos\varphi\,\mathrm{d}\varphi = \cos\varphi + \varphi\sin\varphi - 1$$

$$\mathrm{I}_8 = \int_0^{\varphi} \varphi\sin^2\varphi\,\mathrm{d}\varphi = \frac{\varphi^2}{4} - \frac{1}{4}\left\{\varphi\sin 2\varphi + \frac{1}{2}(\cos 2\varphi - 1)\right\}$$

$$\mathrm{I}_9 = \int_0^{\varphi} \varphi\cos^2\varphi\,\mathrm{d}\varphi = \frac{\varphi^2}{4} + \frac{1}{4}\left\{\varphi\sin 2\varphi + \frac{1}{2}(\cos 2\varphi - 1)\right\}$$

$$\mathrm{I}_{10} = \int_0^{\varphi} \varphi\cos\varphi\sin\varphi\,\mathrm{d}\varphi = -\frac{1}{4}\left(\varphi\cos 2\varphi - \frac{1}{2}\sin 2\varphi\right)$$

$$\mathrm{I}_{11} = \int_0^{\varphi} \varphi^2\sin\varphi\,\mathrm{d}\varphi = -\varphi^2\cos\varphi + 2(\varphi\sin\varphi + \cos\varphi - 1)$$

$$\mathrm{I}_{12} = \int_0^{\varphi} \varphi^2\cos\varphi\,\mathrm{d}\varphi = \varphi^2\sin\varphi + 2(\varphi\cos\varphi - \sin\varphi)$$

$$\mathrm{I}_{13} = \int_0^{\varphi} \varphi^2\sin^2\varphi\,\mathrm{d}\varphi = \frac{\varphi^3}{6} - \frac{1}{4}\left(\varphi^2\sin 2\varphi + \varphi\cos 2\varphi - \frac{1}{2}\sin 2\varphi\right)$$

$$\mathrm{I}_{14} = \int_0^{\varphi} \varphi^2\cos^2\varphi\,\mathrm{d}\varphi = \frac{\varphi^3}{6} + \frac{1}{4}\left(\varphi^2\sin 2\varphi + \varphi\cos 2\varphi - \frac{1}{2}\sin 2\varphi\right)$$

$$\mathrm{I}_{15} = \int_0^{\varphi} \varphi^2\sin\varphi\cos\varphi\,\mathrm{d}\varphi = -\frac{1}{4}\left\{\varphi^2\cos 2\varphi - \varphi\sin 2\varphi - \frac{1}{2}(\cos 2\varphi - 1)\right\}.$$